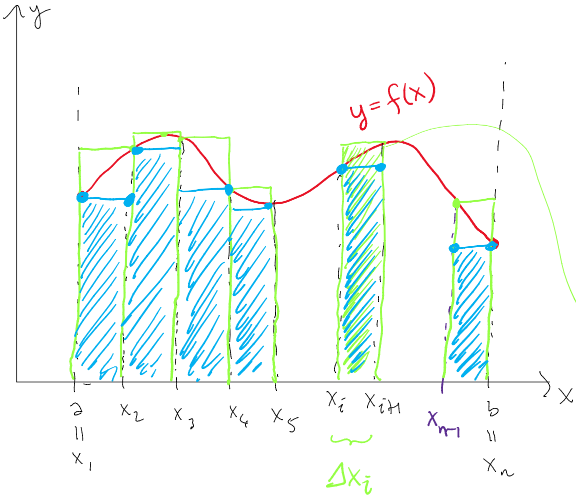
Motivation: How do we find the area of "the region under $y=f(x)$ over $[a,b]$"?



Let f be continuous over $[a,b]$. Given a partition of $[a,b]$, that is, $P=\{x_1,x_2,x_3,\dots,x_n\}$ with $a=x_1<x_2<x_3<\dots<x_n=b$, we define the **upper Riemann sum of f with respect to P** as

$$\mathcal{U}(f,P)=\sum_{i=1}^{n-1}\underbrace{\max_{x_i\le x\le x_{i+1}} f(x)}_{\text{the height of the rectangle}}\cdot\underbrace{\Delta x_i}_{\substack{x_{i+1}-x_i \\ \text{the width of the rectangle}}}$$

this exists because f is continuous on $[x_i,x_{i+1}]$

and **the lower Riemann sum of f with respect to P** as

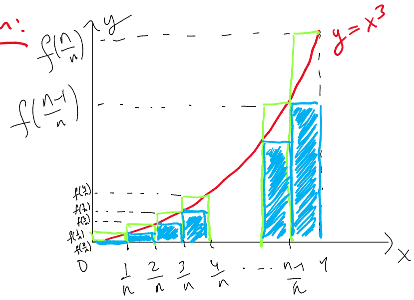
$$L(f,P)=\sum_{i=1}^{n-1}\underbrace{\min_{x_i\le x\le x_{i+1}} f(x)\cdot\Delta x_i}_{\substack{\text{the area of the} \\ i\text{-th blue rectangle}}}.$$

SIGNED

Note that $L(f,P)\le\mathcal{U}(f,P)$.

Example: Find the lower and upper Riemann sums of $f(x)=x^3$ over the interval $[0,1]$ with respect to the partition P_n consisting of n intervals of equal length, that is, $P_n=\{0,\frac{1}{n},\frac{2}{n},\frac{3}{n},\dots,\frac{n-1}{n},1\}$.

Solution:



$$\mathcal{U}(f,P_n)=\text{the sum of the areas of the green rectangles}=f\left(\tfrac{1}{n}\right)\tfrac{1}{n}+f\left(\tfrac{2}{n}\right)\cdot\tfrac{1}{n}+\dots+f\left(\tfrac{n}{n}\right)\cdot\tfrac{1}{n}$$

$$=\sum_{i=1}^{n}f\left(\tfrac{i}{n}\right)\cdot\tfrac{1}{n}=\sum_{i=1}^{n}\frac{i^3}{n^3}\cdot\frac{1}{n}$$

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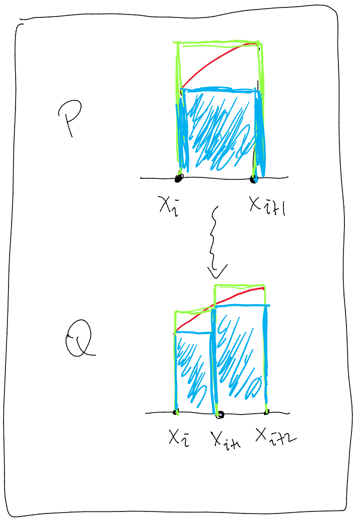
$$=\sum_{i=0}^{n-1}f\left(\tfrac{i}{n}\right)\cdot\tfrac{1}{n}=\sum_{i=0}^{n-1}\frac{i^3}{n^3}\cdot\frac{1}{n}$$

Given partitions P and Q of $[a,b]$, we say that **Q is a refinement of P** if $P\subseteq Q$, that is, every point of P is also in Q. Observe that if Q is a refinement of P, then



$$L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P)$$

We say that f is integrable over $[a,b]$ if there exists a number I and a sequence of partitions $P_1 \subseteq P_2 \subseteq P_3 \subseteq \ldots$ such that $I = \lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} L(f,P_n)$

$$L(f,P_1) \le L(f,P_2) \le \ldots \to I \leftarrow \ldots \le U(f,P_2) \le U(f,P_1)$$

In this case, we say that the definite integral of f over $[a,b]$ is I and write

$$I = \int_a^b f(x)\,dx$$

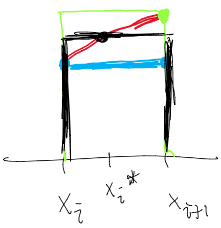
upper limit
integral sign
dummy variable
integrand
lower limit

Geometrically speaking, $\int_a^b f(x)\,dx$ gives us the SIGNED area of the region between $y=f(x)$ and the x-axis over $[a,b]$.

Theorem: If f is continuous over $[a,b]$, then f is integrable over $[a,b]$.

General Riemann Sums: While constructing the upper and lower Riemann sums, if one uses an arbitrary "sample" point x_i^* in $[x_i, x_{i+1}]$ instead of the points which make f attain its max and min values, one can define general Riemann sums.

$$R(f,P,\{x_1^*, x_2^*, \ldots, x_{n-1}^*\}) = \sum_{i=1}^{n-1} f(x_i^*)\,\underline{\underline{\Delta x_i}}$$



Clearly $L(f,P) \le R(f,P,\{x_i^*\}) \le U(f,P)$.

Fact: If f is integrable over $[a,b]$, then

$$\int_a^b f(x)\,dx = \lim_{\substack{n(P)\to\infty \\ \|P\|\to 0}} R(f,P,\{x_i^*\})$$

for all choices of sample points where $n(P)$ denotes the number of points in P and $\|P\|$ denotes $\max_{1\le i\le n(P)-1} \Delta x_i$

Example: Find $\int_0^1 x^3\,dx$ using the definition of the definite integral.

Solution: Recall that we computed the upper and lower Riemann sums of $f(x)=x^3$ with respect to $P_n=\left\{\frac{0}{n},\frac{1}{n},\dots,\frac{n}{n}\right\}$ before. Taking the limits of these we get that

$$\lim_{n\to\infty} L(f,P_n) = \lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{i^3}{n^4} = \lim_{n\to\infty}\frac{1}{n^4}\sum_{i=0}^{n-1} i^3 = \lim_{n\to\infty}\frac{1}{n^4}\left(\frac{(n-1)n}{2}\right)^2 = \underline{\underline{\frac{1}{4}}}$$

$$\boxed{\sum_{i=0}^{k} i^3 = \left(\frac{k(k+1)}{2}\right)^2}$$

$$\lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^3}{n^4} = \lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n} i^3$$

$$= \lim_{n\to\infty}\frac{1}{n^4}\left(\frac{n(n+1)}{2}\right)^2 = \frac{1}{4}$$

Because we have a sequence of partitions $\{P_n\}$ such that $\lim\limits_{n\to\infty} U(f,P_n) = \lim\limits_{n\to\infty} L(f,P_n) = \frac{1}{4}$, we have that $\int_0^1 x^3\,dx = \frac{1}{4}$

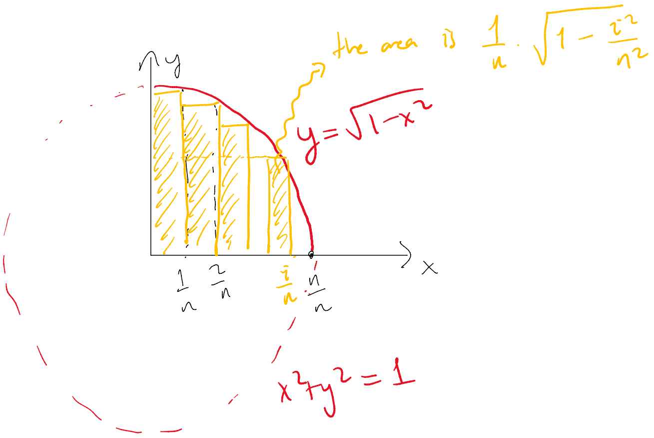
Fact: That "$P_1 \subseteq P_2 \subseteq P_3 \subseteq \dots$" in the definition of integrability can actually be dropped.

Example: Find $\displaystyle\lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{\sqrt{n^2-i^2}}{n^2}\right) = \lim_{n\to\infty}\left(\frac{\sqrt{n^2-1^2}}{n^2}+\frac{\sqrt{n^2-2^2}}{n^2}+\dots\dots+\frac{\sqrt{n^2-(n-1)^2}}{n^2}+\frac{\sqrt{n^2-n^2}}{n^2}\right)$

Solution:
$$\lim_{n\to\infty}\sum_{i=1}^{n}\underbrace{\frac{1}{n}}_{\Delta x_i}\cdot\underbrace{\frac{1}{n}\sqrt{n^2-i^2}}_{f(x_i^*)} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\sqrt{\frac{n^2-i^2}{n^2}} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\sqrt{1-\frac{i^2}{n^2}}$$

$$= \lim_{n\to\infty} L(f,P_n) \text{ where}$$

$f(x)=\sqrt{1-x^2}$ and $P_n=\left\{\frac{0}{n},\frac{1}{n},\dots,\frac{n}{n}\right\}$. As f is integrable over $[0,1]$, we get that



$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\sqrt{n^2-i^2}}{n^2} = \lim_{n\to\infty} L(f,P_n)$$

$$= \int_0^1\sqrt{1-x^2}\,dx$$

$$= \text{the quarter of the area of the unit disc} = \frac{\pi}{4}$$